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PREDICTION OF FAILURE IN NOTCHED INFINITE STRIPS BY THE COHESIVE CRACK MODEL

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Abstract—The potential energy principle of the cohesive crack model is discussed under the condition that the loading device has finite compliance. The relation between crack instability and the second-order variance of the potential energy is examined. With linear softening law, a simple peak load solution is derived from the singularity condition of the potential energy. The obtained formulation is then applied to infinite strips with either central cracks or edge cracks loaded by remote uniform tension. The convergence of the cohesive crack model to linear elastic fracture mechanics solution is demonstrated. It is emphasized that although the concept of failure by crack instability is not ubiquitous, it can provide a common ground to unify linear and nonlinear fracture mechanics as well as tensile strength theory.

1. INTRODUCTION

In the current framework of solid mechanics, a solid is considered to be failed either by excessive deformation, or by a loss of stability as in the example of Euler's strut, or by breakage. In the latter case, when material is elastic and nonlinear hardening deformation (such as plastic flow) is insignificant, failure by breakage is either studied by conventional (tensile) strength theory, or by fracture mechanics. Both theories assume that the breakage of a solid can be associated with certain constant critical quantities which in turn can be extracted from the deformation and stress fields in the solids. In this perspective, fracture mechanics and strength theories are similar in nature, except that the critical quantities used in fracture mechanics are size dependent, while the critical quantities in ordinary strength theories are size independent.

It is generally accepted that while fracture mechanics is suitable for the prediction of rupture load when the crack size is large (compared to an inherent material length), strength theories should be used when the crack size is small. Furthermore, between these two extremes, there should be a nonlinear fracture mechanics that can provide a transition from a strength theory to fracture mechanics. It should be remembered that the process of cracking has to be accompanied by stress softening and deformation localization. Such softening and localization is the major source of nonlinearity for a class of materials, which can be classified as quasi-brittle. Griffith (1920) realized that the effect of such softening and localization was not considered in his theory, and he provided an argument on why such softening behavior can be neglected for the case of brittle fracture:

"...The molecular attraction across such a crack must be small except very near its ends; it may therefore be said that the application of the mathematical theory of elasticity on the basis that the crack is assumed to be a traction-free surface, must give the stresses correctly at all points of the body, with the exception of those near the ends of the crack. In a sufficiently large crack the error in the strain energy so calculated must be negligible."

However, in later studies of fracture mechanics, the softening and localization behavior is still not considered, even when the zone of crack formation is no longer small and negligible. Irwin (1948, 1961) is responsible for advocating the idea that such a nonlinear zone may be taken care of by modifying the concept of surface energy and using an effective crack length. In recent developments the hardening part of nonlinearity is included in constitutive laws. Research has usually focused on devising better fracture criteria that can

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somehow characterize nonlinear deformation around the crack tip. Since Griffith's argument can no longer be applied in these cases due to the presence of a considerable zone of crack formation, the validity of neglecting the softening effect in studies of nonlinear fracture mechanics for quasi-brittle material should be seriously questioned.

An alternative approach in which attraction is included in the equilibrium equations was forwarded by Barenblatt (1962). Among his many discoveries, Barenblatt demonstrated that when attraction is considered, a fracture mechanics without stress singularity can be established; furthermore, if the zone of attraction is vanishingly small, this theory predicts the same rupture load as Griffith theory. However, Barenblatt did not consider the case where process zone (that is, the zone of attraction, the crack formation zone) is not small. On the other hand, although Dugdale's cohesive crack model (1960) has no restriction on process zone size, the feature of stress softening is not captured in his assumption of constant cohesion. Based on finite element technique, Hillerborg *et al.* (1976) developed a numerical method to calculate crack formation and propagation. Softening cohesion law is formally introduced as a constitutive relation for crack formation. For each given process zone length, the load and load–line deflection can be determined by requiring that stress at the crack tip be equal to tensile strength. The most interesting feature is that the rupture load can now be naturally defined as the peak value of the load and load–line deflection curve. Hence, constant critical quantities are no longer needed as in the conventional nonlinear fracture mechanics.

Hillerborg's cohesive crack model was summarized as a potential energy principle advanced by Li and Liang (1992a) for the first time. As will be briefly reviewed in the next section, Griffith's concept of energy balance is extended to include effective crack propagation in the new formulation. In particular, the rupture load of the cohesive crack model is identified with the critical condition of the second-order variation of the potential energy. This discovery seems to have its far-reaching implication in the study of solid mechanics. In fact, it demonstrated for the first time that failure by breakage can also be perceived as a stability problem of solid mechanics, provided that proper constitutive laws are included in the mathematical modeling.

When linear-softening cohesion law is used, such a theory was immediately applied to solve the peak load of 3-point beams (Li and Liang, 1992a), under certain structural mechanics type assumptions on crack opening profile, a very successful analytic solution was obtained. In the same line, a numerical method was developed by Li and Liang (1992b) to solve the peak load without utilizing any assumption on crack opening profile. Using an eigenvalue problem introduced therein, it was shown that under the critical condition, the peak load can be determined without even referring to the crack tip criterion. Later on, such a numerical method was reformulated in the space of continuous functions (Li and Liang, 1992c). With this method, the peak load of the Griffith problem, that is, a finite crack in an infinite plate under remote uniform tension, was solved in the whole range. Numerically it was confirmed that the stability theory will converge to strength theory in the small size limit, and will approach the mechanics of brittle fracture in the large size limit. Thus it is demonstrated that the stability concept can serve as a theoretical foundation for the breakage failure of solids, both strength theory and mechanics of brittle fracture can be obtained as the limiting cases of the stability theory.

In this paper, the behavior of peak load will be studied for an infinite strip under uniform tension with central cracks or symmetrically located edge cracks. Due to the interaction between crack and boundary, many new features of the peak load solution are revealed.

2. THEORY OF CRACK STABILITY

As was shown by Li and Liang (1992a), the cohesive crack model can be represented by a potential energy principle. Under the condition of proportional and monotonic loading, the system shown in Fig. 1 can be assigned a potential energy Π for a given crack length a as follows:

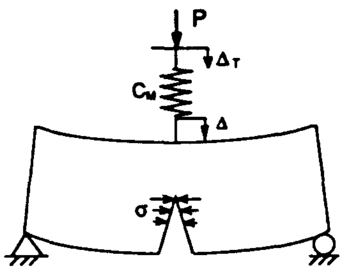

Fig. 1. Schematics of an elastic body with a cohesive crack, the loading device has finite compliance.

$$\Pi(u_i, P, a) = \int_V W(\varepsilon_{ij})\,\mathrm{d}V - P\int_{A_T} b_i u_i\,\mathrm{d}A - \frac{C_M}{2}P^2 + \int_{A_P}\Phi(w)\,\mathrm{d}A, \tag{1}$$

where W is the strain energy density function defined on body V; u_i is the admissible displacement field of the system, and ε_{ij} its corresponding strain field; b_i is the given boundary force distribution defined on A_T, and P is the loading parameter; C_M is the compliance of loading devices; Φ is called a surface potential defined in the part of boundary A_P called the process zone in which cohesive forces act across the crack surfaces; w is the crack separation displacement (normal displacement discontinuity across crack surfaces). The surface potential Φ can be defined for any given cohesion law $\sigma = f(w)$ as:

$$\Phi(w) = \int_0^w f(v)\,\mathrm{d}v. \tag{2}$$

The above definition is a straightforward generalization of surface potential when surface force varies with surface displacement. When the cohesion law is linear-softening up to a threshold value w_c, its corresponding surface potential can be expressed as:

$$\Phi(w) = f_t\int_0^w\left(1-\frac{v}{w_c}\right)\mathrm{d}v = f_t\left(w-\frac{w^2}{2w_c}\right),\quad 0 \leqslant w \leqslant w_c, \tag{3}$$

where f_t is the tensile strength of the material. When the separation w approaches w_c, the cohesive stress σ decreases to zero. Therefrom, Φ becomes a constant.

The displacement equilibrium of the system is defined by equating to zero the first-order variation of the potential Π with respect to displacement:

$$0 = \Pi_u\cdot\delta u_i = \int_V \sigma_{ij}\delta\varepsilon_{ij}\,\mathrm{d}V - P\int_{A_T} b_i\delta u_i\,\mathrm{d}A - \int_{A_P} f(w)\delta w\,\mathrm{d}A. \tag{4}$$

In addition to the displacement equilibrium as described by (4), there is also the concept of crack equilibrium. For a given crack length a, the load P is to be determined by the following crack equilibrium equation. In Hillerborg's model, crack equilibrium is expressed as the condition when the tensile stress at crack tip is equal to tensile strength of the material. The explicit formulation of the crack equilibrium condition in the following form is due to Li and Liang (1992a):

$$0 = \Pi_a = \frac{\partial\Pi}{\partial a} = \frac{\partial}{\partial a}\left(\int_V W\,\mathrm{d}V - P\int_{A_T} b_i u_i\,\mathrm{d}A - \frac{C_M}{2}P^2\right) + \frac{\partial}{\partial a}\int_{A_P}\Phi(w)\,\mathrm{d}A. \tag{5}$$

The first three terms in the parentheses on the right-hand side of eqn (5) can be recognized as the energy release rate; whereas the last term is the energy needed to create a unit area in the process zone and therefore can be referred to as the energy consumption rate. In Griffith theory and Barenblatt theory, the energy consumption rate is assigned a critical value which is considered to be a material property, and eqn (5) is used to determine the critical load. In the cohesive crack model, however, the energy consumption rate is not a constant but deformation dependent. Equation (5) only determines the equilibrium load, not necessarily the critical load. The critical load has to be determined by a stability condition on the potential energy.

When the second order variations of energy Π is not singular, that is, when the corresponding quadratic form will not become zero unless the stress field as well as crack propagation increment is identically zero, then according to the well-known implicit function theorem, eqns (4) and (5) allow the displacement u_i and crack length a to be solved as functions of the loading parameter P, that is, $a = a(P)$ and $u_i = u_i(P)$. After substituting these solutions into the two equilibrium equations, eqns (4) and (5) become identically zero. Differentiation of these equilibrium equations with respect to P can be carried out to obtain the path derivative:

$$\Pi_{uu}\dot{u}_i + \Pi_{ua}\dot{a} + \Pi_{uP} = 0, \tag{6a}$$

$$\Pi_{au}\dot{u}_i + \Pi_{aa}\dot{a} + \Pi_{aP} = 0, \tag{6b}$$

where a dot above a quantity denotes a derivative of the quantity with respect to P. It is to be recalled that the integral in the second term in eqn (1) is the generalized displacement Δ conjugate to P, the partial derivative of the potential energy with respect to loading parameter P can be expressed as

$$\Pi_P = \frac{\partial \Pi}{\partial P} = -\Delta - C_M P = -\Delta_T. \tag{7}$$

The derivative of the total conjugate displacement Δ_T with respect to the loading parameter P is

$$\frac{\mathrm{d}\Delta_T}{\mathrm{d}P} = \frac{\mathrm{d}}{\mathrm{d}P}(-\Pi_P) = -\Pi_{uP}\dot{u}_i. \tag{8}$$

Combining eqns (6a, b) and (8), the reciprocal slope of the P–Δ_T curve can be expressed as:

$$\frac{\mathrm{d}\Delta}{\mathrm{d}P} = \Pi_{uu}\dot{u}_i\dot{u}_j - (\Pi_{aa})_{\Delta_T}\dot{a}^2, \tag{9}$$

where Π_{aa} with subscript Δ_T is the partial derivative of Π_a with respect to a, but under the condition that the total deflection Δ_T is kept constant. The equation can be written as

$$(\Pi_{aa})_{\Delta_T} = \left(\frac{\partial \Pi_a}{\partial a}\right)_{\Delta_T} = \Pi_{aa} + \Pi_{aP}\frac{\mathrm{d}P}{\mathrm{d}a}. \tag{10}$$

Similar to what was indicated by Hutchinson and Paris (1979), at the moment of crack instability, the above equation must become zero. However, in this formulation, there is no need to assume the existence of the R-curve as a material property. As a matter of fact, the R-curve can be calculated from the cohesive crack model, as was first demonstrated by Foote *et al.* (1986), and later by Liang and Li (1991). Furthermore, it is shown that the R-curve is size dependent. As has been pointed out by Li and Liang (1992c), the R-curve has to be size dependent so that in the large size limit it can become flat. Otherwise, the cohesive crack model will not converge to linear elastic fracture mechanics.

In the case of dead weight loading, the reciprocal slope $\mathrm{d}\Delta_T/\mathrm{d}P$ presented in eqn (9) is infinite at the moment of crack instability, consequently, the form Π_{uu} has to be singular, in the sense that Π_{uu} has a zero eigenvalue. In the case of displacement controlled loading, the crack instability corresponds to snap-back in the $P-\Delta_T$ curve, hence $d\Delta_T/dP = 0$. Under such a condition, Π_{uu} is no longer singular. In matrix terminology, only a submatrix of Π_{uu} will become singular. However, the discussion of the computational method for the case of snap-back is out of the scope of this paper.

Π_{aa} and Π_{uu} are conceptually different. Π_{aa} directly determines the stability property of crack propagation, whereas Π_{uu} characterizes the stability property of the whole system. The foregoing discussion shows that if crack propagation is the only mechanism that will cause the system to lose stability, Π_{uu} may also be used to characterize the stability of crack propagation under the condition of dead weight loading. This important linkage between the singularity condition of Π_{uu} and the instability of crack propagation was first noticed by Li and Liang (1992a). It is also to be noted that the key element that causes Π_{uu} to lose its positive definiteness is strain softening, or in other words, the fact that cohesion law is a decreasing function of crack separation displacement. If the cohesion law is constant (Dugdale, 1960), then Π_{uu} will never lose its positive definiteness. Finally, a word of caution is necessary. The condition $\Pi_{uu} = 0$ is only a necessary condition for a system to lose stability, just as $\Pi_{aa} = 0$ is only a necessary condition for crack instability. If the structure is such that Π_{aa} is always non-negative, then the singularity condition of Π_{uu} should not be used to predict crack instability.

3. THEORY OF PEAK LOAD DETERMINATION

From now on, it is assumed that the cohesion law is linear, and the surface potential Φ is represented by eqn (3). In order to facilitate the discussion, the following notations are introduced. Let $(f_i, u_i)_A$ be the corresponding virtual work on the part of the boundary denoted by A. Note that A can be A_T for the boundary where force is prescribed, or the boundary A_P, the process zone. When $A = A_P$, it is to be understood that only the virtual work of normal forces on the normal displacement discontinuity across the crack surfaces is counted. Finally $a(u_i, v_i)$ denotes the virtual elastic work of deformation state u_i on displacement v_i, which is the result of the first-order variation of strain energy with respect to displacement. It is noted that all these forms are symmetrical in the sense that an interchange of the positions of their two arguments will not change the value of the forms.

The displacement equilibrium equation (4) can be rewritten, using these notations, as:

$$a(u_i, v_i) - P(h_i, v_i)_{A_T} + f_t(1, v_i)_{A_P} - \frac{f_t}{w_c}([u_i], [v_i])_{A_P} = 0, \tag{11}$$

where v_i is any virtual displacement, by which it means that it satisfies the homogeneous displacement boundary condition on A_u. $[\cdot]$ is the displacement discontinuity across the crack surfaces. It is known from previous discussions that at peak load the bilinear form Π_{uu} is singular, mathematically it is the same to state that there is a nontrivial solution u_i^* such that

$$a(u_i^*, v_i) = \frac{f_t}{w_c}([u_i^*], [v_i])_{A_P}, \quad \forall v_i, \tag{12}$$

where v_i is the virtual displacement. The terms on the right-hand side of the equation are obviously related to the quadratic term of the surface potential Φ, as shown in eqn (3). It can be seen that in the domain V the solution u_i^* satisfies the homogeneous equilibrium equations. On the boundary, it satisfies the homogeneous stress and displacement condition. In the process zone the boundary condition is such that the normal stress component $\sigma_n = [u_n] f_t w_c$ and the shear stress component $\tau = 0$, where $[u_n]$ is the discontinuity of the normal displacement u_n in the process zone.

Since it is assumed that the equilibrium solution u_i of eqn (11) satisfies the homogeneous condition on A_u, the solution u_i can be used as a virtual displacement in eqn (12). On the other hand, the non-trivial solution u_i^* of eqn (12) can be used as a virtual displacement in eqn (11). Due to the symmetrical property, the first and last terms in eqn (11) cancel out, hence the peak load P_{cr} can be obtained as:

$$P_{cr} = \frac{f_i(1, u_i^*)_{A_p}}{(h_i, u_i^*)_{A_r}}. \tag{13}$$

It is to be noted that the existence of a non-trivial solution u_i^* to eqn (12) depends upon the crack length a (or process zone length, to be more precise), but is independent of the equilibrium solution. For given structural dimensions, to find the critical process zone length such that Π_{uu} is singular is computationally expensive, since the mesh has to be updated to accommodate the changing process zone length. On the other hand, for a given process zone length, the singularity condition can be used to determine the critical dimensions of the structure. When size effect is one of the major objectives, the latter computational approach is very attractive due to its simplicity. Such a computational approach has been used by Li and Liang (1992c) to solve the Griffith problem.

4. BASIC EQUATIONS FOR NOTCHED INFINITE STRIPS

For an infinite strip under uniform tension in infinity with symmetrically located cracks as shown in Fig. 2, the equilibrium equation can be expressed as an integral equation using the dislocation density function $G(x) = \partial v(x, 0)/\partial x$ as the unknown. $v(x, y)$ is the vertical displacement component. According to Gupta and Erdogan (1974) the integral equation can be written as

$$\int_a^b \left[\frac{1}{t-x} + \frac{1}{t+x} + k(x, t) - k(x, -t)\right] G(t)\, dt = -\frac{1+\kappa}{4\mu} \pi\sigma(x), \tag{14}$$

where μ is the shear modulus. $\kappa = 3-4\nu$ for plane strain condition and $\kappa = (3-\nu)/(1+\nu)$ for plane stress condition, and ν is the Poisson's ratio. The kernel $k(x, t)$ can be expressed as

$$k(x, t) = \int_0^\infty K(x, t, s)\, e^{-(2h-t)s}\, ds, \tag{15}$$

and the integrand can be written as

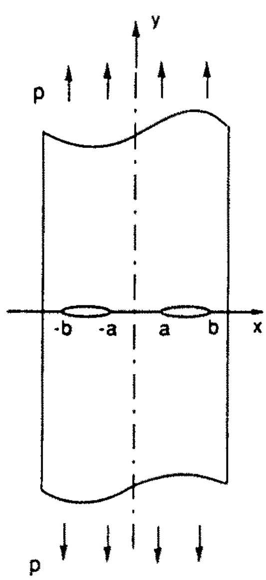

Fig. 2. An infinite strip under remote uniform tension with symmetrically located cracks.

$$K(x,t,s) = -\{[1+(3+2sh)\,\mathrm{e}^{-2sh}]\cosh(sx) - 2sx\sinh(sx)\,\mathrm{e}^{-2sh} - [2sx\sinh(sx) + (3-2sh+\mathrm{e}^{-2sh})\cosh(sx)][1-2s(h-t)]\}/(1+4sh\,\mathrm{e}^{-2sh}-\mathrm{e}^{-4sh}). \quad (16)$$

It is noted that the function $K(x, t, s)$ actually has a singularity of the first order at $s = 0$, but it will be cancelled out in the calculation of $k(x,t)-k(x, -t)$. In this paper, we will only study two limiting cases. The first case is the central crack, which corresponds to letting $a \to 0$. The second case is the double-notch configuration, which amounts to letting $b \to h$. As was pointed out by Gupta and Erdogan (1974), for the case where $b \to h$, the kernel $k(x, t)$ becomes unbounded when both x and $t \to h$. In order to avoid numerical difficulties, the terms contained in the numerator of $K(x,t,s)$ that do not converge to zero fast enough when $s \to \infty$ will be subtracted from, and then added to $K(x,t,s)$. After this modification, the integral equation can be written as

$$\int_a^h \left[\frac{1}{t-x} + \frac{1}{t+x} + k_f(x,t) + k_s(x,t)\right] G(t)\,\mathrm{d}t = -\frac{1+\kappa}{4\mu}\pi\sigma(x), \quad (17)$$

where $k_f(x, t)$ is a Fredholm kernel defined as

$$k_f(x,t) = -k(x,t) + \int_0^\infty [K(x,t,s) - K_\infty(x,t,s)]\,\mathrm{e}^{-(2h-x-t)s}\,\mathrm{d}s. \quad (18)$$

The expression for $k(x, t)$ and $K(x, t, s)$ has been defined in eqns (15) and (16), respectively. The expression K_∞ is the term in $K(x, t, s)$ that causes $k(x, t)$ to be unbounded at $x, t = h$, and is found to be

$$K_\infty = \mathrm{e}^{\prime\prime}\{-2+[3(h-t)+(h-x)]s-2s^2(h-t)(h-x)\}. \quad (19)$$

However, K_∞ can be integrated analytically to yield the definition for k_s as shown below:

$$k_s(x,t) = \int_0^\infty K_\infty(x,t,s)\,\mathrm{e}^{-(2h-x-t)s}\,\mathrm{d}s = \frac{-2}{2h-x-t} + \frac{3(h-t)+(h-x)}{(2h-x-t)^2} - \frac{4(h-t)(h-x)}{(2h-x-t)^3}. \quad (20)$$

It is seen that k_s indeed has strong singularity when $x, t = h$. Such a singularity is necessary in order for the solution $G(x)$ to be free of power singularity at $x = h$. Although the selection of $K_\infty(x, t, s)$ is arbitrary to some extent, the resulting $k_s(x, t)$ should always be determined via eqn (20) from the selected K_∞. For an unknown reason, the corresponding expressions given in Gupta and Erdogan's paper (1974) do not seem to be related in this way.

The function $\sigma(x)$ is the stress component normal to the crack surfaces. In accordance with eqn (12), the stress is determined by the linear term in the cohesion law with the process zone, that is, $\sigma(x) = 2f_t v(x)/w_c$. The factor 2 is introduced because v is only one half of the crack opening displacement.

In order to simplify the notation, let's first introduce a linear coordinate transformation so that in both cases the crack tip corresponds to $x = 1$. On the other hand, $x = 0$ will correspond to the edge point for the case of edge cracks, and correspond to the central point of the crack for the case of a central crack. Furthermore, let β be a real number between 0 and 1 such that the process zone lies between β and 1. With such notations, the integral equation can be expressed as

$$\frac{4\mu}{1+\kappa}\frac{1}{\pi}\int_0^1 M(x,t)G(t)\,\mathrm{d}t = H(x-\beta)2f_t\frac{v(x)}{w_c}, \quad (21)$$

where $H(x-\beta)$ is the Heaviside step-function such that if $x \leqslant \beta$, the function is zero, otherwise it is 1. The integral kernel $M(x, t)$ should take its corresponding form according to eqns (17) or (14) for different crack configurations.

The numerical solution uses the following representation of the unknown function $G(x)$:

$$G(x) = \frac{\varphi(x)}{\sqrt{1-x^2}}, \tag{22}$$

so that the new unknown function $\varphi(x)$ is free of the inverse-square-root singularity in the crack tip $x = 1$. The displacement v can be represented by the new unknown function through the following integration:

$$v(x) = -d\int_x^1 \frac{\varphi(t)}{\sqrt{1-t^2}}\,dt, \tag{23}$$

where d is the total crack length. In the case of an edge crack, d is the length of one crack; in the case of a central crack, d is one half of the crack length. The quadrature points t_k and the collocation points x_r are defined according to Gupta and Erdogan (1974) as:

$$t_k = \cos\left(\frac{2k-1}{4n+2}\pi\right), \qquad x_r = \cos\left(\frac{r\pi}{2n+1}\right), \tag{24}$$

where n is the total number of quadrature points, k and r vary from 1 to n. The discretized integral equation can be written as

$$\frac{1}{2n+1}\frac{4\mu}{1+\kappa}\sum_{k=1}^{n} M(x_r, t_k)\varphi(t_k) = H(x_r-\beta)f_t\frac{2d}{w_c}\sum_{k=1}^{s} B_{rk}\varphi(t_k), \tag{25}$$

where the coefficients B_{kr} are calculated from eqn (23). Trapezoidal formula or Simpson's formula is used to interpolate the function φ among the quadrature points t_k. The upper limit s on the right-hand side of eqn (25) depends upon the value of β such that x_s is the closest to β. Actually in this computational scheme, β is chosen to be equal to x_r successively for $r = 1, 2, 3, \ldots$, as will be seen in the following discussion. Equation (25) can be further written in a matrix form as

$$[A]\{\varphi\} = 2d^*[B]_s\{\varphi\}. \tag{26}$$

The subscript s of $[B]$ means that all elements starting with row $s+1$ and thereafter are zero. Such a structure is caused by the step-function $H(x-\beta)$. The quantity d^* is called the nondimensional crack length and is defined as d/l_{ch}. The characteristic length l_{ch} of given material is defined as

$$l_{ch} = \frac{4\mu w_c}{(1+\kappa)f_t}, \tag{27}$$

which is basically the same as the one used by Hillerborg *et al.* (1976). However, the above definition recognizes the difference between the plane stress condition and the plane strain condition.

Equation (26) is the discretized form of the singularity condition for Π_{uu}. In particular, the singularity condition can be satisfied by either adjusting the process zone length for given d^*, or by adjusting d^* for a given process zone length. In the latter approach, (26) can be viewed as an eigenvalue problem with $2d^*$ as its smallest eigenvalue. The nontrivial solution u_i^* can now be recognized as its corresponding eigenfunction.

It is noted that β is related to the crack length by the relation $\beta = d_0/d$, where d_0 is the initial crack length. d_0 is considered as a given data, while d varies with β. Consequently, in all previous expressions (including those integral kernels which have had the linear coordinate transformation of their arguments x and t, since the transformation depends upon d), d should be replaced by d_0/β. With these notations, our calculation can be described as being given β to find d_0 such that (26), the singularity condition of Π_{mn}, can be satisfied.

After the nontrivial solution is found through eqn (26), the corresponding displacement can be obtained by the following equation:

$$\{v^*\} = [B]\{\varphi\}. \tag{28}$$

The peak load can be determined by eqn (13). Notice that the tension in the infinite can be equivalently transferred onto crack surface as a uniform pressure, the function b_i is 1 in the direction of crack opening and zero otherwise. Hence eqn (13) can be expressed as

$$\frac{p_{cr}}{f_t} = \frac{\int_\beta^1 v^*(x)\,dx}{\int_0^1 v^*(x)\,dx}, \tag{29}$$

where the proper numerical quadrature rule should be applied to evaluate these two integrations. Now for a given value of β, the peak load can be determined by eqn (29), and the corresponding size d_0 can be determined from the eigenvalue of eqn (26).

5. NUMERICAL RESULTS

In order that the obtained peak load results can be compared with the linear elastic fracture mechanics theory, the following quantity is defined

$$\eta = \frac{p_{cr}}{f_t}\sqrt{d_0^*}. \tag{30}$$

In linear elastic fracture mechanics, η is proportional to the stress intensity factor, and is a constant within a group of geometrically similar specimens. In the cohesive crack model, however, η varies with specimen size even in a geometrically similar group. Such a scaling effect by the cohesive crack model is due to the fact that the size of the process zone is no longer negligible. The variation of η is the reason why linear elastic fracture mechanics should not be applied.

There are two different ways to calibrate a stress intensity factor from the peak load solution. One way is to use the initial crack length as is implied in eqn (30), the other is to use the total crack length d. Using total crack length is a practice similar to the effective crack length method. It is usually assumed that the effective crack length method can be used to compensate the nonlinear effect in the crack tip. Our calculation suggests that the effect of such a modification is quite complicated. In Figs 3–5 the results are shown of

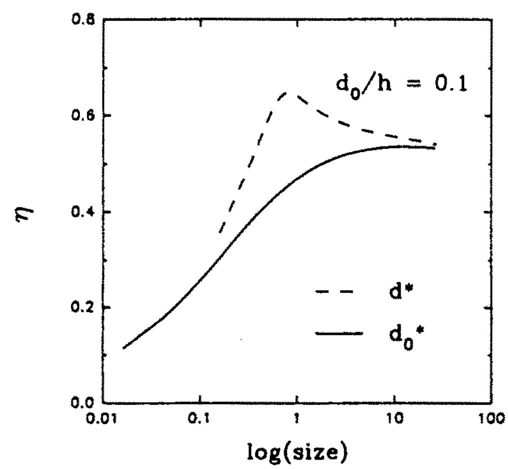

Fig. 3. η values of central cracks with $d_0/h = 0.1$.

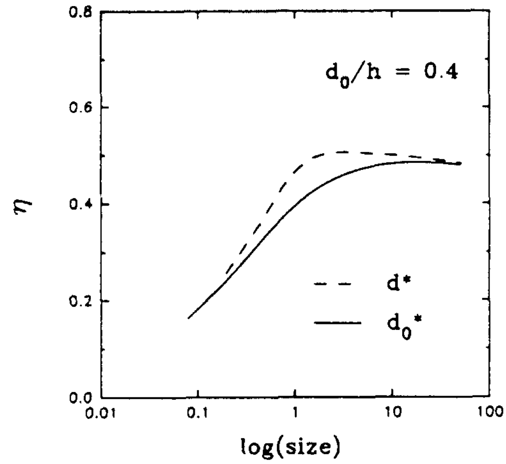

Fig. 4. η values of central cracks with $d_0/h = 0.4$.

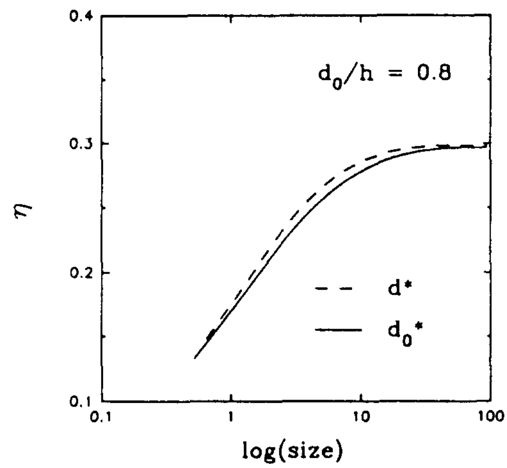

Fig. 5. η values of central cracks with $d_0/h = 0.8$.

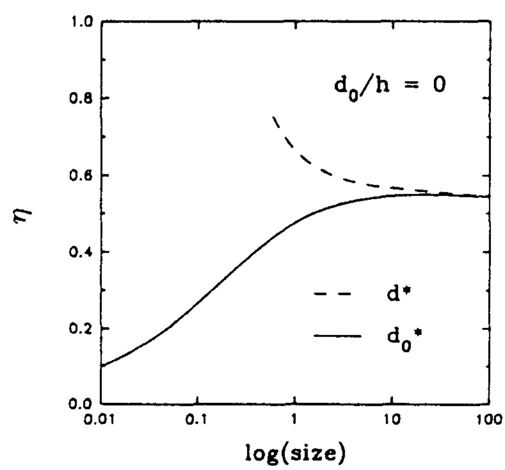

Fig. 6. η values of central cracks in an infinite plane.

central cracks with different crack-width ratios d_0/h. The abscissa labeled "size" is either the total crack length d or the initial crack length d_0, depending upon the curve it represents. It can be seen that the difference between the two calibrating methods is quite dramatic when the crack-width ratio is small, but become closer to each other when the notch becomes deeper. Similar trends can be seen in edge crack specimens shown in Figs 7-9. Such a variance in effective-crack-length calibration can be attributed to the interaction between crack and boundary. When there is no boundary present, such as a central crack in an infinite plate, the η-values are plotted in Fig. 6. In this case, the η value according to

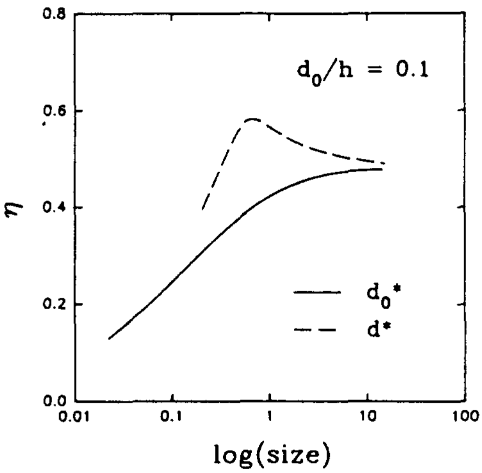

Fig. 7. η values of edge cracks with $d_0/h = 0.1$.

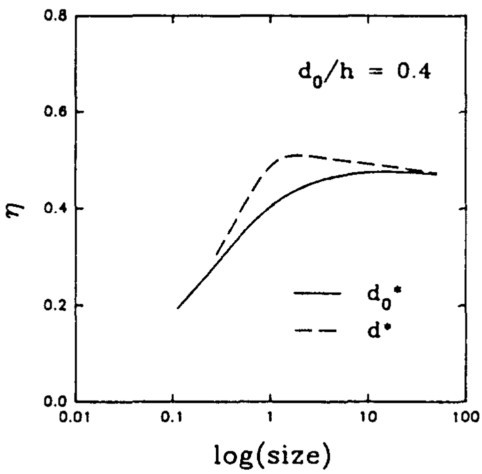

Fig. 8. η values of edge cracks with $d_0/h = 0.4$.

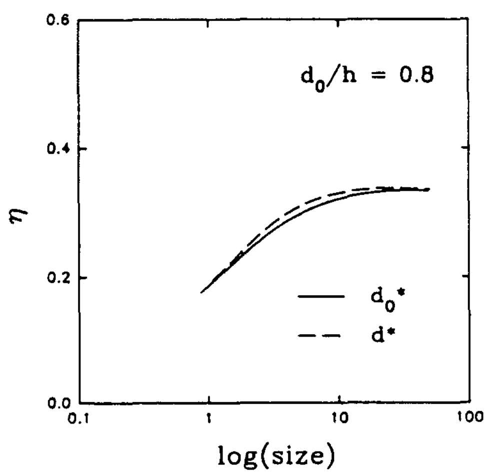

Fig. 9. η values of edge cracks with $d_0/h = 0.8$.

the effective-crack-length method is a monotonically decreasing function of size, quite different to the results in the strips. However, in all these cases, the η value calculated using the initial crack length d_0 is always an increasing function of specimen size.

Comparing the results of central cracks in an infinite strip with the results of a central crack in an infinite plane, it can be seen that the calculation can be carried out for very small size d_0^* in an infinite plane, while there is an obvious lower limit for d_0^* in an infinite strip. In the case of an infinite plane, the peak load p_{cr} converges to the tensile strength f_t when d_0^* aproaches zero. In the case of an infinite strip, the smallest d_0^* obtainable from the calculation is finite, and the corresponding peak load is significantly less than $(1-d_0/h)f_t$. A question naturally arises: if the size d_0^* is less than the lower limit, is the cohesive crack model still applicable?

Physically, it is easy to understand that when the size approaches zero, the stress state becomes increasingly uniform, consequently, the strength theory can be recovered from the cohesive crack model. However, in the case of an infinite strip, the strength theory limit cannot be reached. Since the width of a strip is finite, under remote tension, the process zone may need to be stretched to be very long before the peak load can be reached. When the required process zone length exceeds the available ligament length, the strip is failed before the load–deflection curve can have a horizontal tangent. In other words, the strip is failed by crack traversal, not by crack instability. The theory presented in this paper can only be applied to find the load of the instability crack. That is why there will be a nonzero lower limit d_0^*. However, that is not to say that the cohesive crack model cannot be applied in this case. For example, Hillerborg's method can still be used to find the load when a strip is failed by crack traversal.

When the size d_0^* is large, the difference between the two calibrating methods becomes insignificant. Because the relative process zone length of the specimen becomes very small, the relative difference between d_0 and d becomes very small. The calculated η value approaches a constant, which is determined by material properties as well as the ratio d_0/h. It is noted that the case of a central crack in an infinite plane can be treated as a central crack in an infinite strip with $d_0/h = 0$. The Griffith solution can be written as

$$p_G = \sqrt{\frac{8\mu}{1+\kappa}\frac{2\gamma}{\pi d_0}}. \tag{31}$$

Because the surface energy 2γ is equal to the total work needed to create a new crack surface, which is $f_t w_c/2$ in the cohesive crack model with linear softening, the above equation can be equivalently written as

$$\frac{p_G}{f_t} = \frac{1}{\sqrt{\pi d_0^*}} = \frac{0.564}{\sqrt{d_0^*}}. \tag{32}$$

That is to say, the η value calculated according to eqn (30) approaches 0.564 in the large size limit for central cracks in an infinite plane. In the case of a central crack in an infinite strip, however, there will be a correction factor $f(d_0/h)$ such that in the large size limit, the η value calculated according to eqn (30) will be equal to $0.564/f(d_0/h)$. Similarly, for the edge crack configuration, there will also be a correction factor, denoted as $f_2(d_0/h)$ such that in the large size limit, $\eta = 0.564/f_2(d_0/h)$. Evidently, these correction factors are the stress intensity factors in their respective crack tips under unit load, and their values can be found, for instance, in the paper by Delale and Erdogan (1977) and Gupta and Erdogan (1974). On the other hand, if the calculated η is multiplied by f and f_2, respectively, then the modified η value will all converge to the same value, that is 0.564, in the large size limit. These results are shown in Figs 10–11 for central-crack and edge-crack configurations.

It should be pointed out that the numerical method employed herein only has a linear convergence rate in the large size range. For instance, when there are less than, say, 20 quadrature nodes within the process zone (that is, between β and 1), the obtained peak load is underestimated. If the number of quadrature nodes n is doubled, the error will only

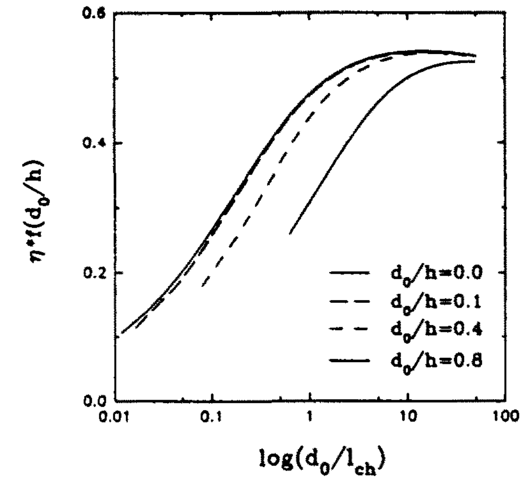

Fig. 10. Modified η values of different central-cracked strips.

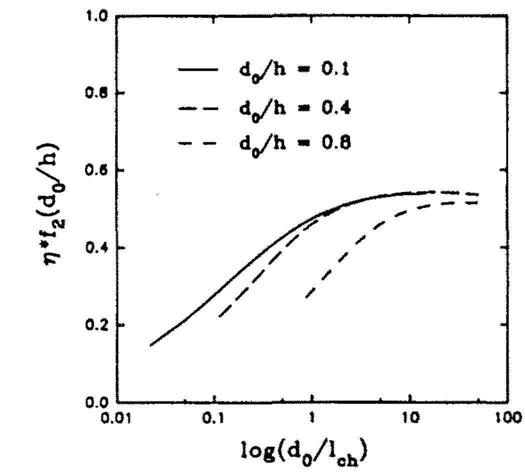

Fig. 11. Modified η values of different edge-cracked strips.

be reduced by one half. Consequently, the theoretical value cannot be touched but only approached. When n is in the region of 200, the largest η (after correction by multiplying by f or f_2) is only about 95% of 0.564. Such accuracy can hardly be considered as totally satisfactory. However, other than in the large size range, the accuracy is excellent, judging from the convergence of the numerical results. Since the solution in the large size limit can be known by a linear elastic fracture mechanics solution, such a deficiency in the numerical procedure does not seem that unbearable. Nonetheless, it is hoped that a better numerical approach will be taken on in a future study.

6. CONCLUSIONS

In this paper, the potential energy principle of the cohesive crack model is discussed under the assumption that the loading devices have finite compliance. The peak load solution is derived based on the crack instability condition. It is demonstrated that under dead weight loading, crack instability can be formulated as a singularity condition of Π_{uu}, the second-order variance of potential energy with respect to displacement. Singular integral equation is used to express the singularity condition of Π_{uu} in infinite strips with either central cracks or edge cracks. The obtained peak load solution is expressed in the form of η as a function of specimen size. The behavior of such solutions in both small size range and large size range is discussed.

The phenomena of failure are diverse and complicated, by no means is the stability theory discussed herein meant to embrace all the failure mechanisms. Even in the case of simple tension we can distinguish between failure by crack instability and failure by crack traversal. However, the concept of crack instability does seem to provide a common ground to understand nonlinear fracture mechanics and linear fracture mechanics, as well as simple strength theory. The method of associating a critical quantity with failure only works in limiting cases. In a general situation, the concept that identifies failure with instability seems to be more fundamental than the concept of critical quantity in understanding failure phenomena.

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REFERENCE

Barenblatt, G. I. (1962). The mathematical theory of equilibrium cracks in brittle fracture. *Adv. Appl. Mech.* **7**, 55–129.

Delale, F. and Erdogan, F. (1977). The problem of internal and edge cracks in an orthotropic strip. *J. Appl. Mech.* **44**, 237–242.

Dugdale, D. S. (1960). Yielding of steel sheets containing slits. *J. Mech. Phys. Solids* **8**, 100–104.

Foote, R. M. L., Mai, Y.-W. and Cotterell, B. (1986). Crack growth resistance curves in strain-softening materials. *J. Mech. Phys. Solids* **34**(6), 593–607.

Griffith, A. A. (1920). The phenomenon of rupture and flow in solids. *Phil. Trans. R. Soc.* **A221**, 163–198.

Gupta, G. D. and Erdogan, F. (1974). The problem of edge cracks in an infinite strip. *ASME J. Appl. Mech.* **41**, 1001–1006.

Hillerborg, A., Modeer, M. and Petersson P.-E. (1976). Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. *Cement and Concrete Res.* **6**, 773–782.

Hutchinson, J. W. and Paris, P. C. (1979). *Stability Analysis of J-controlled Crack Growth*. ASTM STP 688, American Society for Testing and Materials, Philadelphia.

Irwin, G. R. (1948). Fracture dynamics. In *Fracture of Metals*, pp. 147–166. Trans. ASM, Cleveland, Ohio.

Irwin, G. R. (1961). Plastic zone near a crack and fracture toughness. Sagamore Research Conference Proceedings, Vol. 4.

Hutchinson, J. W. and Paris, P. C. (1979). *Stability analysis of J-controlled Crack Growth*. ASTM STP 688, American Society for Testing and Materials, Philadelphia.

Li, Y. N. and Liang, R. Y. (1992a). Stability theory of the cohesive crack model. *J. Engng Mech. ASCE* **118**(3).

Li, Y. N. and Liang, R. Y. (1992b). Theory and algorithm of peak load determination in the linear fictitious crack model. *Int. J. Fract.* (submitted).

Li, Y. N. and Liang, R. Y. (1992c). The theory of boundary eigenvalue problem in the cohesive crack model and its application. *J. Mech. Phys. Solids* (accepted).

Liang, R. Y. and Li, Y. N. (1991). A study of size effect in concrete using fictitious crack model. *J. Engng Mech, ASCE*, 1631–1651.